MATH 3235 Probability Theory

09/08/2022

Probability Spaces $(\Omega, \mathcal{F}, \mathbb{P})$

Random Variables X (discrete)

Bernoulli

Binomial

Geometric

Hypergeometric

Poisson

You have a fuel pump on the HW.

# of cars that arrives in an hour has a poisson distribution w. par λ

Each car that arrives has a prop 0.1 of needing service.

Distribution of arriving cars that need service?

Y number of cars arriving

X " " " " need service

$$p_Y(y) = e^{-\lambda} \frac{\lambda^y}{y!}$$

$$p_X(x) = \mathbb{P}(X = x) = \sum_{y \geq x} \mathbb{P}(X = x \mid Y = y)\, \mathbb{P}(Y = y)$$

$$\mathbb{P}(X = x \mid Y = y) = \binom{y}{x} 0.1^x\, 0.9^{y-x}$$

$$p_X(x) = \sum_{y \geq x} \binom{y}{x} 0.1^x\, 0.9^{y-x} e^{-\lambda} \frac{\lambda^y}{y!} =$$

$$\binom{y}{x} = \frac{y!}{x!(y-x)!}$$

$$\sum_{y \geq x} \frac{1}{(y-x)!} \frac{1}{x!} 0.1^x\, 0.9^{y-x} e^{-\lambda} \lambda^y =$$

$$\frac{e^{-0.1\lambda}}{x!} (0.1\lambda)^x \left[\sum_{y-x \geq 0} \frac{1}{(y-x)!} (0.9\lambda)^{y-x} e^{-0.9\lambda} \right]$$

$z = y - x$

$$e^{-0.9\lambda} \sum_{z \geq 0} \frac{1}{z!} \left((0.9)\lambda \right)^{z} = 1$$

I Know That $\sum_{x \geq 0} p_X(x) = 1$.

$$p_X(x) = C e^{-0.1\lambda} \frac{\lambda^{x}}{x!}$$

$$\sum_{x \geq 0} p_X(x) = C \implies C = 1.$$

Expectation

X is a discrete r.v.

$p(x)$

$$\mathbb{E}(X) = \sum_x x\, p(x)$$

I have an experiment whose result is described by X.

I repeat it many times, N.

$x_1, x_2 \ldots x_N$ Sample of size N.

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i =$$

$$= \frac{1}{N} \sum_{\substack{y \\ \text{possible values}}} (\#\text{times the result was } y)\, y$$

$$= \frac{1}{N} \sum_y |\{ i \mid x_i = y \}|\, y$$

$$\frac{|\{ i \mid x_i = y \}|}{N} \xrightarrow[N \to \infty]{} p(y)$$

$\mathbb{E}(X)$ "Sample mean of an extremely large sample!"

$$\mathbb{E}(X) = \sum_{x \in \operatorname{Im}(X)} p(x)\, x$$

Bernoulli:

$$\mathbb{E}(X) = p \quad \text{if } X \sim \operatorname{Ber}(p)$$

$$\mathbb{E}(X) = 0 \cdot p(0) + 1 \cdot p(1) = p$$

Binomial

$X \sim \operatorname{Bin}(n, p)$

$$\mathbb{E}(X) = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x}$$

$$x\binom{n}{x} = x \frac{n!}{x!\,(n-x)!} = \frac{n\,(n-1)!}{(x-1)!\,(n-x)!} =$$

$$n\binom{n-1}{x-1}$$

$$\mathbb{E}(X) = \sum_{x=1}^{n} n \binom{n-1}{x-1} p^x q^{n-x} =$$

$$= np \sum_{y=0}^{n-1} \binom{n-1}{y} p^y q^{n-1-y} =$$

$y = x - 1$

$$= np$$

N is poissonian par λ

$$\mathbb{E}(N) = \sum_{n \geq 0} n \frac{\lambda^n}{n!} e^{-\lambda} =$$

$$\lambda \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} e^{-\lambda} = \lambda$$

$Bin(n, p)$ n large

$p = \frac{\lambda}{n}$

$$np \longrightarrow \lambda$$

Hypergeometric

N S in the pop.
M F in the pop.

$p = N/(N+M)$

n extractions

X = number of Ss

$$\mathbb{E}(X) = np = \frac{nN}{N+M}$$

Geometric p

$$p(x) = q^{x-1} p \qquad x \geq 1$$

$$\mathbb{E}(X) = \sum_{x \geq 1} x q^{x-1} p$$

$$\sum_{x \geq 1} x q^{x-1} = \sum_{x \geq 1} \left(\frac{d}{dq} q^x \right) =$$

$$= \frac{d}{dq} \sum_{x \geq 1} q^x = \frac{d}{dq} \left(\frac{q}{1-q} \right) =$$

$$= \frac{1}{1-q} + \frac{q}{(1-q)^2} = \frac{1-q+q}{(1-q)^2} =$$

$$= \frac{1}{(1-q)^2}$$

$$\mathbb{E}(X) = \frac{p}{(1-q)^2} = \frac{1}{p}$$

If I have a r.v. X and a function $f : \mathbb{R} \to \mathbb{R}$ Then $Y = f(X)$ is a discrete r.v.

If $X \sim Ber(p)$

$f(x) = ax + b$

$Y = f(X)$ $\quad Y = \begin{cases} b & \text{prob } q \\ a+b & \text{prob } p \end{cases}$

Any r.v. Y That Take only

Two value can be written as a linear function of a Ber. r.v. X.

X^2, $|X|$, e^X, ...

if $Y = f(X)$, $p_Y(y)$?

$y \in \mathrm{Im}(Y)$ I need to find all $\omega \in \Omega$ such that $Y(\omega) = y$. $\Rightarrow$ find all ω such that

$$f(X(\omega)) = y$$

$$\Downarrow$$

$$\omega \mid X(\omega) = x \text{ and } f(x) = y$$

$$p_Y(y) = \sum_{\omega \mid Y(\omega) = y} \mathbb{P}(\{\omega\}) = \sum_{x \mid f(x) = y} \sum_{\omega \mid X(\omega) = x} \mathbb{P}(\{\omega\})$$

$$= \sum_{x \mid f(x) = y} P_X(x)$$

$$P_Y(y) = \sum_{x \mid f(x) = y} P_X(x)$$

X with values in $\mathbb{Z}$

$Y = X^2$

$$P_Y(y) = P_X(\sqrt{y}) + P_X(-\sqrt{y})$$

$$P_Y(4) = P_X(2) + P_X(-2).$$